Th 1: If M is a linear subspace of a Hilbert space H, then show that —

$$M \text{ is closed} \Leftrightarrow M = M^{\perp\perp}$$

Pf: Let M be a subspace of H.
and $M = M^{\perp\perp}$.
Then to show that M is closed.

Now
$$M = M^{\perp\perp} = (M^{\perp})^{\perp}$$
Since orthogonal complement of a subspace is a closed subspace,

$\therefore$ $(M^{\perp})^{\perp}$ is a closed subspace of H.

$\Rightarrow$ $M = M^{\perp\perp}$ is a closed sub-space of H.

Converse Suppose that M is a closed subspace of H. Then to prove that
$$M = M^{\perp\perp}$$
We know that $M \subset M^{\perp\perp}$.

Suppose $M \neq M^{\perp\perp}$

$M^{\perp\perp}$ is itself a Hilbert space and M is a proper closed subspace of $M^{\perp\perp}$.

$\therefore \exists$ a non-zero vector z_0 in $M^{\perp\perp}$ s.t. $z_0 \perp M$ or $z_0 \in M^{\perp}$.

Now $z_0 \in M^{\perp}$ and $z_0 \in M^{\perp\perp}$

$\Rightarrow z_0 \in M^{\perp} \cap M^{\perp\perp}$ —(1)

Since $M^{\perp}$ is a subspace of H, ∴ we have

$$M^{\perp} \cap M^{\perp\perp} = \{0\} \rightarrow (2)$$

∴ From (1) & (2), we have

$$z_0 = 0$$

which contradicts that z_0 is a non zero vector. ∴ we must have

$$M = M^{\perp\perp}$$

(Proved)

Th:- Projection Theorem

If M is a closed linear subspace of a H.S. H, then $H = M \oplus M^{\perp}$ → direct sum

Pf:- If M is a subspace of a H.S. H then we know that $M \cap M^{\perp} = \{0\}$.

∴ To show that $H = M \oplus M^{\perp}$, we need to chk if $H = M + M^{\perp}$

Now we know that $M^{\perp}$ is a closed subspace of H. Given that M is a closed subspace of H. ∴ $M + M^{\perp}$ is also a closed subspace of H.

Put $N = M + M^{\perp} \rightarrow (1)$

∴ $M \subset N$ & $M^{\perp} \subset N$.

∴ we have

$$N^{\perp} \subset M^{\perp} \text{ \& } N^{\perp} \subset M^{\perp\perp}$$

or $N^{\perp} \subset M^{\perp} \cap M^{\perp\perp}$

or $N^{\perp} \subset M^{\perp} \cap M$, as $M^{\perp\perp} = M$

or $N^{\perp} \subset M \cap M^{\perp} = \{0\}$

$\therefore$ $N^{\perp} = \{0\}$

$\Rightarrow (N^{\perp})^{\perp} = \{0\}^{\perp}$

$\Rightarrow N^{\perp\perp} = H$, $\because \{0\}^{\perp} = H$ $\rightarrow (2)$

Since $N = M + M^{\perp}$ is a closed subspace of H, $\therefore$ $N^{\perp\perp} = N$

$\therefore$ From (2), we get –

$N = H$

$\Rightarrow H = N = M + M^{\perp}$

i.e. $H = M + M^{\perp}$

& $M \cap M^{\perp} = \{0\}$

$\therefore$ $H = M \oplus M^{\perp}$

(Proved)